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Evolution families and nonuniform spectrum

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Abstract. We give a complete description of all possible forms of the nonuniform spectrum for an evolution family on a Banach space. Moreover, for each form we provide an explicit example of a nonautonomous differential equation on $l^2(\mathbb{N})$ whose evolution family has that spectrum. As an application, we show that the asymptotic behavior persists under sufficiently small nonlinear perturbations, in the sense that the lower and upper Lyapunov exponents of the nonlinear dynamics are in the same connected component of the nonuniform spectrum.

Keywords: nonuniform hyperbolicity, spectrum, Lyapunov regularity.

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1 Introduction

For an evolution family on a Banach space, we give a complete description of all possible forms of the nonuniform spectrum. This notion of spectrum is inspired on the one introduced by Sacker and Sell in [12] in terms of uniform exponential dichotomies. Instead, we consider nonuniform exponential dichotomies with an arbitrarily small nonuniform part, for which the conditional stability, although exponential, need not be uniformly exponential on the initial time. We emphasize that these exponential dichotomies are very common in the context of ergodic theory—in strong contrast, the notion of uniform exponential dichotomy is much more restrictive. In particular, almost all trajectories with nonzero Lyapunov exponents of a measure-preserving flow give rise to a linear variational equation admitting a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. Our results can also be considered a contribution to the theory of nonuniform hyperbolicity, which is an important tool in the study of stochastic behavior. We refer the reader to [1] for a detailed exposition of the theory, which goes back to landmark works of Oseledets [8] and particularly Pesin [9].

Given an evolution family $T(t,s)$ of linear operators acting on a Banach space, its *nonuniform spectrum* is the set Σ of all numbers $a \in \mathbb{R}$ such that the evolution family $e^{-a(t-s)}T(t,s)$ does not admit a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. Our main aim is to describe the structure of the nonuniform spectrum (see Theorem 2.2):

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Main Theorem. *The nonuniform spectrum Σ is either $\emptyset$, $\mathbb{R}$, a finite union of disjoint closed intervals (possibly unbounded), or there exists numbers*

$$b_1 \geq a_1 > b_2 \geq a_2 > b_3 \geq a_3 > \cdots$$

such that

$$\Sigma = I_1 \cup \bigcup_{n=2}^{\infty} [a_n, b_n] \quad \text{or} \quad \Sigma = I_1 \cup \bigcup_{n=2}^{\infty} [a_n, b_n] \cup (-\infty, a_\infty],$$

where $I_1 = [a_1, b_1]$ or $I_1 = [a_1, +\infty)$, respectively if $a_n \to -\infty$ or $a_n \to a_\infty$.

Moreover, we describe how the nonuniform spectrum relates to certain invariant subspaces (see Theorems 2.7 and 2.8). In particular, we show that each trajectory of the evolution family has lower and upper Lyapunov exponents inside the same connected component of the nonuniform spectrum. For related work we refer the reader to [3,6,13].

In addition, the asymptotic behavior persists under sufficiently small nonlinear perturbations, in the sense that the lower and upper Lyapunov exponents of the nonlinear dynamics belong to the same connected component of the nonuniform spectrum (see Theorem 2.10). More precisely, consider a nonzero global solution $x(t)$ of the nonlinear equation

$$x(t) = T(t,s)x(s) + \int_s^t T(t,\tau) f(\tau, x(\tau))\, d\tau$$

such that

$$a_p \leq \liminf_{t \to +\infty} \frac{1}{t} \log\|x(t)\| \leq \limsup_{t \to +\infty} \frac{1}{t} \log\|x(t)\| \leq \sup \Sigma$$

and

$$\lim_{t \to +\infty} \int_t^{t+1} e^{\delta\tau} \frac{\|f(\tau, x(\tau))\|}{\|x(\tau)\|}\, d\tau = 0$$

for some $\delta > 0$. Then there exists $i \in \{1, \ldots, p\}$ such that

$$a_i \leq \liminf_{t \to +\infty} \frac{1}{t} \log\|x(t)\| \leq \limsup_{t \to +\infty} \frac{1}{t} \log\|x(t)\| \leq b_i.$$

A related result was established by Coppel in [4] for perturbations of a linear differential equation with constant coefficients. Corresponding results for perturbations of autonomous delay equations were established by Pituk [10,11] (for values in a finite-dimensional space) and by Matsui, Matsunaga and Murakami [7] (for values in a Banach space).

Finally, for each possible form of the nonuniform spectrum Σ we provide an explicit example of an evolution family on $l^2(\mathbb{N})$ having that spectrum (see Section 3).

2 Nonuniform spectrum

2.1 Preliminaries

Let $B(X)$ be the set of all bounded linear operators acting on a Banach space X. A family $T(t,s)$, for $t, s \in \mathbb{R}$ with $t \geq s$, of linear operators in $B(X)$ is called an *evolution family* if:

1. $T(t,t) = \mathrm{Id}$ for $t \in \mathbb{R}$;

2. $T(t,s)T(s,\tau) = T(t,\tau)$ for $t, s, \tau \in \mathbb{R}$ with $t \geq s \geq \tau$.

We say that an evolution family $T(t,\tau)$ admits a *nonuniform exponential dichotomy with an arbitrarily small nonuniform part* or simply a *nonuniform dichotomy* if:

1. there exist projections $P_t\colon X \to X$ for $t \in \mathbb{R}$ with $\dim \operatorname{Ker} P_t < +\infty$ satisfying
$$T(t,s)P_s = P_t T(t,s) \tag{2.1}$$
for $t \geq s$ such that the map
$$T(t,s)|\operatorname{Ker} P_s\colon\ \operatorname{Ker} P_s \to \operatorname{Ker} P_t$$
is invertible;

2. there exist $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that
$$\|T(t,s)P_s\| \leq De^{-\lambda(t-s)+\varepsilon|s|} \quad \text{for } t \geq s \tag{2.2}$$
and
$$\|T(t,s)Q_s\| \leq De^{-\lambda(s-t)+\varepsilon|s|} \quad \text{for } t \leq s, \tag{2.3}$$
where $Q_t = \mathrm{Id} - P_t$ and
$$T(t,s) = (T(s,t)|\operatorname{Ker} P_t)^{-1}\colon\ \operatorname{Ker} P_t \to \operatorname{Ker} P_s$$
for $t < s$.

The sets $\operatorname{Im} P_t$ and $\operatorname{Im} Q_t$ are called, respectively, *stable* and *unstable spaces* of the nonuniform dichotomy. We note that the hypothesis that the unstable spaces are finite-dimensional already appeared for example in [5,13].

Proposition 2.1. *For each $t \in \mathbb{R}$, we have*
$$\operatorname{Im} P_t = \left\{ v \in X : \sup_{s \geq t} \|T(s,t)v\| < +\infty \right\}$$
and $\operatorname{Im} Q_t$ consists of all vectors $v \in X$ for which there exists a function $x\colon (-\infty, t] \to X$ such that $x(t) = v$, $x(t_1) = T(t_1,t_2)x(t_2)$ for $t \geq t_1 \geq t_2$ and $\sup_{s \leq t}\|x(s)\| < +\infty$.

Proof. By (2.2) we have
$$\sup_{s \geq t} \|T(s,t)v\| < +\infty \tag{2.4}$$
for $v \in \operatorname{Im} P_t$. On the other hand, if $v \in X$ satisfies (2.4), then it follows from (2.2) that
$$\sup_{s \geq t} \|T(s,t)Q_t v\| < +\infty. \tag{2.5}$$
By (2.3), for $s \geq t$ we have
$$\|Q_t v\| \leq De^{-\lambda(s-t)+\varepsilon|s|}\|T(s,t)Q_t v\|.$$
Whenever $Q_t v \neq 0$, taking $\varepsilon < \lambda$ we obtain
$$\sup_{s \geq t} \|T(s,t)Q_t v\| = +\infty,$$
which contradicts to (2.5). Hence, $Q_t v = 0$ and $v \in \operatorname{Im} P_t$.

Now take a vector $v \in \operatorname{Im} Q_t$ and consider the function $x\colon (-\infty, t] \to X$ defined by $x(s) = T(s,t)v$ for $s \le t$. Then $x(t_1) = T(t_1,t_2)x(t_2)$ for $t \ge t_1 \ge t_2$ and it follows from (2.3) that $\sup_{s\le t}\|x(s)\| < +\infty$. On the other hand, there exists no $v \in \operatorname{Im} P_t \setminus \{0\}$ for which there is a function $x\colon (-\infty, t] \to X$ as in the proposition. Indeed, it follows from (2.1) and (2.2) that

$$\|v\| = \|T(t,s)P_s x(s)\| \le De^{-\lambda(t-s)+\varepsilon|s|}\|x(s)\|$$

for $s \le t$. Taking $\varepsilon < \lambda$ yields that $\sup_{s\le t}\|x(s)\| = +\infty$. □

The *nonuniform spectrum* of an evolution family $T(t,s)$ is the set Σ of all numbers $a \in \mathbb{R}$ such that the evolution family $T_a(t,s) = e^{-a(t-s)}T(t,s)$ does not admit a nonuniform dichotomy. For each $a \in \mathbb{R}$ and $t \in \mathbb{R}$, let

$$S_a(t) = \left\{ v \in X : \sup_{s \ge t} \left(e^{-a(s-t)}\|T(s,t)v\|\right) < +\infty \right\}$$

and let $U_a(t)$ be the set of all vectors $v \in X$ for which there exists a function $x\colon (-\infty, t] \to X$ such that $x(t) = v$, $x(t_1) = T(t_1,t_2)x(t_2)$ for $t \ge t_1 \ge t_2$ and

$$\sup_{s \le t} \left(e^{-a(s-t)}\|x(s)\|\right) < +\infty.$$

We note that if $a < b$, then

$$S_a(t) \subset S_b(t) \quad \text{and} \quad U_b(t) \subset U_a(t)$$

for $t \in \mathbb{R}$. By Proposition 2.1, if $a \in \mathbb{R} \setminus \Sigma$, then

$$X = S_a(t) \oplus U_a(t) \quad \text{for } t \in \mathbb{R}$$

and the dimensions $\dim S_a(t)$ and $\dim U_a(t)$ are independent of t.

2.2 Main result

The following theorem is our main result. It describes all possible forms of the nonuniform spectrum.

Theorem 2.2. *For an evolution family $T(t,s)$ on a Banach space, one of the following alternatives holds:*

1. $\Sigma = \emptyset$;
2. $\Sigma = \mathbb{R}$;
3. Σ *is a finite union of disjoint closed intervals (possibly unbounded);*
4. $\Sigma = I_1 \cup \bigcup_{n=2}^{\infty}[a_n, b_n]$, *where* $I_1 = [a_1, b_1]$ *or* $I_1 = [a_1, +\infty)$, *for some numbers*
$$b_1 \ge a_1 > b_2 \ge a_2 > b_3 \ge a_3 > \cdots \tag{2.6}$$
with $\lim_{n\to+\infty} a_n = -\infty$;
5. $\Sigma = I_1 \cup \bigcup_{n=2}^{\infty}[a_n, b_n] \cup (-\infty, a_\infty]$, *where* $I_1 = [a_1, b_1]$ *or* $I_1 = [a_1, +\infty)$, *for some numbers as in* (2.6) *with* $a_\infty = \lim_{n\to+\infty} a_n$.

Proof. We first establish some auxiliary results.

Lemma 2.3. *The set $\Sigma \subset \mathbb{R}$ is closed and for each $a \in \mathbb{R} \setminus \Sigma$ we have $S_a(t) = S_b(t)$ and $U_a(t) = U_b(t)$ for all $t \in \mathbb{R}$ and all b in some open neighborhood of a.*

Proof of the lemma. Given $a \in \mathbb{R} \setminus \Sigma$, there exist projections P_t for $t \in \mathbb{R}$ satisfying (2.1), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|e^{-a(t-s)}T(t,s)P_s\| \leq De^{-\lambda(t-s)+\varepsilon|s|}$$

for $t \geq s$ and

$$\|e^{-a(t-s)}T(t,s)Q_s\| \leq De^{-\lambda(s-t)+\varepsilon|s|}$$

for $t \leq s$. Therefore, for each $b \in \mathbb{R}$,

$$\|e^{-b(t-s)}T(t,s)P_s\| \leq De^{-(\lambda-a+b)(t-s)+\varepsilon|s|}$$

for $t \geq s$ and

$$\|e^{-b(t-s)}T(t,s)Q_s\| \leq De^{-(\lambda+a-b)(s-t)+\varepsilon|s|}$$

for $t \leq s$. Hence, $b \in \mathbb{R} \setminus \Sigma$ whenever $|a - b| < \lambda$ and it follows from Proposition 2.1 that $S_b(t) = S_a(t)$ and $U_b(t) = U_a(t)$ for $t \in \mathbb{R}$. $\square$

Lemma 2.4. *Take $a_1, a_2 \in \mathbb{R} \setminus \Sigma$ with $a_1 < a_2$. Then $[a_1, a_2] \cap \Sigma \neq \emptyset$ if and only if $\dim U_{a_1}(t) > \dim U_{a_2}(t)$.*

Proof of the lemma. Assume that $\dim U_{a_1}(t) = \dim U_{a_2}(t)$. Then $U_{a_1}(t) = U_{a_2}(t)$ and $S_{a_1}(t) = S_{a_2}(t)$ for $t \in \mathbb{R}$. Hence, by Proposition 2.1, there exist projections P_t for $t \in \mathbb{R}$ satisfying (2.1), constants $\lambda_1, \lambda_2 > 0$ and for each $\varepsilon > 0$ constants $D_1 = D_1(\varepsilon), D_2 = D_2(\varepsilon) > 0$ such that for $i = 1, 2$ we have

$$\|e^{-a_i(t-s)}T(t,s)P_s\| \leq D_i e^{-\lambda_i(t-s)+\varepsilon|s|} \quad \text{for } t \geq s \tag{2.7}$$

and

$$\|e^{-a_i(t-s)}T(t,s)Q_s\| \leq D_i e^{-\lambda_i(s-t)+\varepsilon|s|} \quad \text{for } t \leq s. \tag{2.8}$$

For each $a \in [a_1, a_2]$, by (2.7) we obtain

$$\|e^{-a(t-s)}T(t,s)P_s\| \leq D_1 e^{-\lambda_1(t-s)+\varepsilon|s|} \quad \text{for } t \geq s$$

and similarly, by (2.8),

$$\|e^{-a(t-s)}T(t,s)Q_s\| \leq D_2 e^{-\lambda_2(s-t)+\varepsilon|s|} \quad \text{for } t \leq s.$$

Taking $\lambda = \min\{\lambda_1, \lambda_2\}$ and $D = \max\{D_1, D_2\}$ yields that $[a_1, a_2] \subset \mathbb{R} \setminus \Sigma$.

For the converse, assume that $\dim U_{a_1}(t) > \dim U_{a_2}(t)$ and let

$$b = \inf\left\{a \in \mathbb{R} \setminus \Sigma : \dim U_a(t) = \dim U_{a_2}(t)\right\}.$$

Since $\dim U_{a_1}(t) > \dim U_{a_2}(t)$, it follows from Lemma 2.3 that $a_1 < b < a_2$. Now assume that $b \notin \Sigma$. Then either $\dim U_b(t) = \dim U_{a_2}(t)$ or $\dim U_b(t) \neq \dim U_{a_2}(t)$. In the first case, by Lemma 2.3, there exists $\varepsilon > 0$ such that $\dim U_{b'}(t) = \dim U_{a_2}(t)$ and $b' \in \mathbb{R} \setminus \Sigma$ for $b' \in (b - \varepsilon, b]$. But this contradicts to the definition of b. In the second case, again by Lemma 2.3, there exists $\varepsilon > 0$ such that $\dim U_{b'}(t) \neq \dim U_{a_2}(t)$ and $b' \in \mathbb{R} \setminus \Sigma$ for $b' \in [b, b + \varepsilon)$, which again contradicts to the definition of b. Hence, $b \in \Sigma$ and $[a_1, a_2] \cap \Sigma \neq \emptyset$. $\square$

Lemma 2.5. *For each $c \notin \Sigma$, the set $\Sigma \cap [c, +\infty)$ is the union of finitely many closed intervals.*

Proof of the lemma. Let
$$d = \dim U_c(t) = \dim \operatorname{Ker} P_t,$$
where P_t are the projections associated to the nonuniform dichotomy of the evolution family $e^{-c(t-s)}T(t,s)$. We assume that $\Sigma \cap [c, +\infty)$ has at least $d+2$ connected components $I_i = [\alpha_i, \beta_i]$, for $i = 1, \ldots, d+2$, where
$$\alpha_1 \le \beta_1 < \alpha_2 \le \beta_2 < \cdots < \alpha_{d+2} \le \beta_{d+2} \le +\infty.$$
For $i = 1, \ldots, d+1$, take $c_i \in (\beta_i, \alpha_{i+1})$. It follows from Lemma 2.4 that
$$d > \dim U_{c_1}(t) > \dim U_{c_2}(t) > \cdots > \dim U_{c_{d+1}}(t),$$
which is impossible. □

Now we assume that Σ is not given by one of the first three alternatives in the theorem and take $c_1 \notin \Sigma$. By Lemma 2.5, the set $\Sigma \cap [c_1, +\infty)$ is the union of finitely many disjoint closed intervals, say $I_1, \ldots, I_k$. We note that $\Sigma \cap (-\infty, c_1) \neq \varnothing$, since otherwise $\Sigma = I_1 \cup \cdots \cup I_k$, which contradicts to our assumption. Moreover, there exists $c_2 < c_1$ such that $c_2 \notin \Sigma$ and $(c_2, c_1) \cap \Sigma \neq \varnothing$. Otherwise, $(-\infty, c_1) \cap \Sigma = (-\infty, a]$ for some $a < c_1$ and thus,
$$\Sigma = (-\infty, a] \cup I_1 \cdots \cup I_k,$$
which again contradicts to our assumption. Proceeding inductively, we obtain a decreasing sequence $(c_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that
$$c_n \notin \Sigma \quad \text{and} \quad (c_{n+1}, c_n) \cap \Sigma \neq \varnothing$$
for $n \in \mathbb{N}$. Now either $\lim_{n \to +\infty} c_n = -\infty$ or $\lim_{n \to +\infty} c_n = a_\infty$ for some $a_\infty \in \mathbb{R}$. In the first case, it follows from Lemma 2.5 that Σ is given by alternative 4. In the second case, it follows from Lemma 2.5 that
$$(a_\infty, \infty) \cap \Sigma = I_1 \cup \bigcup_{n=2}^{\infty} [a_n, b_n],$$
where $I_1 = [a_1, b_1]$ or $I_1 = [a_1, +\infty)$, for some sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ as in (2.6) with $a_\infty = \lim_{n \to +\infty} a_n$. Again by Lemma 2.5, we have $(-\infty, a_\infty] \subset \Sigma$ and so Σ is given by the last alternative. □

The finite-dimensional case is simpler.

Theorem 2.6. *For an evolution family on a finite-dimensional space, the nonuniform spectrum is given by one of the first three alternatives in Theorem 2.2.*

Proof. Assume that the ambient space has dimension d. We will show that Σ is the union of at most $d+1$ disjoint closed intervals. This implies that Σ is never given by the last two alternatives in Theorem 2.2.

Assume that Σ has at least $d+2$ connected components. Then there exist numbers $c_1, \ldots, c_{d+1} \in \mathbb{R} \setminus \Sigma$ such that $c_i < c_{i+1}$ and $(c_i, c_{i+1}) \cap \Sigma \neq \varnothing$ for $i = 1, \ldots, d$. It follows from Lemma 2.4 that
$$d \ge \dim U_{c_1}(t) > \dim U_{c_2}(t) > \ldots > \dim U_{c_{d+1}}(t),$$
which is impossible. □

2.3 Further properties

In this section we assume that Σ is neither $\emptyset$ nor $\mathbb{R}$. Let $(c_k)_k \subset \mathbb{R}$ be a finite or infinite sequence such that $c_k \in (b_{k+1}, a_k)$ for each k, with the numbers a_k and b_k as in (2.6) and define

$$E_k(s) = S_{c_k}(s) \cap U_{c_{k+1}}(s), \qquad k = 1, 2, \ldots$$

Moreover, when $\Sigma \cap \mathbb{R}^+$ is bounded, take $c_0 > b_1$ and define

$$E_0(s) = S_{c_0}(s) \cap U_{c_1}(s).$$

By Lemma 2.4, the subspaces $E_k(s)$ are independent of the numbers c_k.

Theorem 2.7. *Assume that Σ is neither $\emptyset$ nor $\mathbb{R}$. For each $k = 1, 2, \ldots$, $s \in \mathbb{R}$ and $v \in E_k(s) \setminus \{0\}$, we have*

$$\left[\liminf_{t\to+\infty} \frac{1}{t} \log\|T(t,s)v\|, \limsup_{t\to+\infty} \frac{1}{t} \log\|T(t,s)v\|\right] \subset [a_{k+1}, b_{k+1}].$$

When $\Sigma \cap \mathbb{R}^+$ is bounded, this statement also holds for $k = 0$.

Proof. Since $c_k \notin \Sigma$, the evolution family $e^{-c_k(t-s)}T(t,s)$ admits a nonuniform dichotomy and so there exist projections P_t for $t \in \mathbb{R}$ satisfying (2.1), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|T(t,s)P_s\| \le De^{(c_k-\lambda)(t-s)+\varepsilon|s|} \quad \text{for } t \ge s \tag{2.9}$$

and

$$\|T(t,s)Q_s\| \le De^{-(\lambda+c_k)(s-t)+\varepsilon|s|} \quad \text{for } t \le s,$$

where $Q_t = \mathrm{Id} - P_t$. By Proposition 2.1, we have $\operatorname{Im} P_t = S_{c_k}(t)$ for $t \in \mathbb{R}$. Hence, each $v \in E_k(s)$ belongs to $\operatorname{Im} P_s$ and so, by (2.9),

$$\limsup_{t\to+\infty} \frac{1}{t} \log\|T(t,s)v\| \le c_k - \lambda < c_k.$$

Letting $c_k \searrow b_{k+1}$, we obtain

$$\limsup_{t\to+\infty} \frac{1}{t} \log\|T(t,s)v\| \le b_{k+1}.$$

Similarly, since $c_{k+1} \notin \Sigma$, there exist projections P'_t for $t \in \mathbb{R}$ satisfying (2.1), a constant $\mu > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|T(t,s)P'_s\| \le De^{(c_{k+1}-\mu)(t-s)+\varepsilon|s|} \quad \text{for } t \ge s$$

and

$$\|T(t,s)Q'_s\| \le De^{-(\mu+c_{k+1})(s-t)+\varepsilon|s|} \quad \text{for } t \le s, \tag{2.10}$$

where $Q'_t = \mathrm{Id} - P'_t$. By Proposition 2.1, we have $\operatorname{Im} Q'_t = U_{c_{k+1}}(t)$ for $t \in \mathbb{R}$. Hence, each $v \in E_k(s)$ belongs to $\operatorname{Im} Q'_s$ and so, by (2.10),

$$\|v\| \le De^{-(\mu+c_{k+1})(t-s)+\varepsilon|t|}\|T(t,s)v\| \quad \text{for } t \ge s.$$

Taking ε sufficiently small, we obtain

$$\liminf_{t\to+\infty} \frac{1}{t} \log\|T(t,s)v\| \ge \mu + c_{k+1} - \varepsilon > c_{k+1}$$

and letting $c_{k+1} \nearrow a_{k+1}$ yield that

$$\liminf_{t \to +\infty} \frac{1}{t} \log\|T(t,s)v\| \geq a_{k+1}.$$

This completes the proof of the theorem. □

A similar argument yields a corresponding statement for negative time.

Theorem 2.8. *Assume that Σ is neither $\emptyset$ nor $\mathbb{R}$. For each $k = 1, 2, \ldots$, $s \in \mathbb{R}$ and $v \in E_k(s) \setminus \{0\}$, there exists a function $x \colon (-\infty, s] \to X$ such that $x(s) = v$, $x(t_1) = T(t_1, t_2)x(t_2)$ for $s \geq t_1 \geq t_2$ and*

$$\left[\liminf_{t \to -\infty} \frac{1}{t} \log\|x(t)\|, \limsup_{t \to -\infty} \frac{1}{t} \log\|x(t)\|\right] \subset [a_{k+1}, b_{k+1}].$$

When $\Sigma \cap \mathbb{R}^+$ is bounded, this statement also holds for $k = 0$.

The following example illustrates Theorems 2.7 and 2.8.

Example 2.9. Consider the evolution family $T(t,s)$ obtained from the nonautonomous linear equation $x' = A(t)x$ with

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 3t^2 \end{pmatrix}.$$

For each $a > 1$, the evolution family $T_a(t,s) = e^{-a(t-s)}T(t,s)$ admits a nonuniform dichotomy with projections $P_t(x,y) = (x,0)$ (see Example 3.1 below for details). On the other hand, for $a < 1$ the evolution family $T_a(t,s)$ admits a nonuniform dichotomy with projections $P_t = 0$. Clearly, $T_1(t,s)$ does not admit a nonuniform dichotomy and so $\Sigma = \{1\}$.

Now take $c_1 < 1 < c_0$ (which corresponds to take $a_1 = b_1 = 1$). Then

$$E_0(t) = S_{c_0}(t) \cap U_{c_1}(t) = (\mathbb{R} \times \{0\}) \cap \mathbb{R}^2 = \mathbb{R} \times \{0\}$$

and by Theorems 2.7 and 2.8, for $s \in \mathbb{R}$ and $v = (x,0) \in \mathbb{R} \times \{0\}$ with $x \neq 0$, we have

$$\lim_{t \to \pm\infty} \frac{1}{t} \log\|T(t,s)v\| = 1.$$

2.4 Nonlinear perturbations

It turns out that the asymptotic behavior described in Theorem 2.7 persists under sufficiently small nonlinear perturbations. Given an evolution family $T(t,s)$ on a Banach space X, we consider the nonlinear equation

$$x(t) = T(t,s)x(s) + \int_s^t T(t,\tau)f(\tau, x(\tau))\,d\tau \tag{2.11}$$

for some continuous map $f \colon \mathbb{R} \times X \to X$. Repeating arguments in the proof of Theorem 6 in [2] we obtain the following result.

Theorem 2.10. *For an evolution family $T(t,s)$ on a Banach space such that Σ is neither $\emptyset$ nor $\mathbb{R}$, let $x(t)$ be a nonzero global solution of equation (2.11) such that*

$$a_p \leq \liminf_{t \to +\infty} \frac{1}{t} \log\|x(t)\| \leq \limsup_{t \to +\infty} \frac{1}{t} \log\|x(t)\| \leq \sup \Sigma$$

for some integer p and

$$\lim_{t\to+\infty}\int_t^{t+1} e^{\delta\tau}\frac{\|f(\tau,x(\tau))\|}{\|x(\tau)\|}\,d\tau=0$$

for some $\delta>0$. Then there exists $i\in\{1,\dots,p\}$ such that

$$a_i\le\liminf_{t\to+\infty}\frac{1}{t}\log\|x(t)\|\le\limsup_{t\to+\infty}\frac{1}{t}\log\|x(t)\|\le b_i,$$

with the convention that $b_1=+\infty$ when $I_1=[a_1,+\infty)$.

3 Examples

In this section we provide explicit examples of all possible forms of the nonuniform spectrum Σ given by Theorem 2.2. Let $X=\ell^2(\mathbb{N})$ be a separable infinite-dimensional Hilbert space with the orthonormal basis $\{e_1,e_2,\dots\}$.

Example 3.1. Consider the evolution family $T(t,s)$ on X given by

$$T(t,s)e_n=\begin{cases}e^{t^3-s^3}e_1, & n=1,\\ e^{s^3-t^3}e_n, & n\ge 2.\end{cases}$$

It is obtained from the linear equation $x'=A(t)x$, where $A(t)e_1=3t^2e_1$ and $A(t)e_n=-3t^2e_n$ for $n\ge 2$.

We claim that $\Sigma=\emptyset$. We first consider the evolution family $T^1(t,s)=e^{t^3-s^3}$ on $\mathbb{R}$. Given $a\in\mathbb{R}$ and $\lambda>0$, consider the function $g\colon\mathbb{R}\to\mathbb{R}$ given by

$$g(t)=-at+t^3-\lambda t.$$

There exists $C>0$ such that g is increasing on the intervals $(-\infty,-C)$ and $(C,+\infty)$. Hence,

$$e^{-a(t-s)+t^3-s^3+\lambda(s-t)}=e^{g(t)-g(s)}\le 1$$

whenever $t\le s<-C$ or $C<t\le s$. This implies that there exists $D>0$ such that

$$e^{-a(t-s)+t^3-s^3+\lambda(s-t)}\le D$$

for $t\le s$ and so

$$(T^1)_a(t,s)\le De^{-\lambda(s-t)}$$

for $t\le s$. Hence, $(T^1)_a(t,s)=e^{-a(t-s)}T^1(t,s)$ admits a nonuniform dichotomy with projections $P_t=0$. Now we consider the evolution family $T^2(t,s)=e^{s^3-t^3}$. Proceeding as above, one can show that $(T^2)_a(t,s)$ admits a nonuniform dichotomy with projections $P_t=\mathrm{Id}$. Therefore, $T_a(t,s)$ admits a nonuniform dichotomy with projections P_t given by $P_te_1=0$ and $P_te_n=e_n$ for $n\ge 2$.

Example 3.2. Consider the evolution family $T(t,s)$ on X given by

$$T(t,s)e_n=\begin{cases}e^{ct\cos t-cs\cos s-c\sin t+c\sin s}e_1, & n=1,\\ e^{s^3-t^3}e_n, & n\ge 2,\end{cases}$$

where $c > 0$. It is obtained from the linear equation $x' = A(t)x$, where $A(t)e_1 = -ct\sin te_1$ and $A(t)e_n = -3t^2e_n$ for $n \geq 2$.

We claim that $\Sigma = \mathbb{R}$. For this it is sufficient to prove that the nonuniform spectrum of the evolution family

$$T^1(t,s) = e^{ct\cos t - cs\cos s - c\sin t + c\sin s}$$

is $\mathbb{R}$. Take $a \in \mathbb{R}$ and assume that the evolution family $(T^1)_a(t,s)$ admits a nonuniform dichotomy with projections P_t. There are two possibilities: either $P_t = \mathrm{Id}$ for all $t \in \mathbb{R}$ or $P_t = 0$ for all $t \in \mathbb{R}$. In the first case, there exist $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$e^{-a(t-s)}T^1(t,s) \leq De^{-\lambda(t-s)+\varepsilon|s|} \quad \text{for } t \geq s.$$

In particular, for $t = 2l\pi$ and $s = (2l-1)\pi$ with $l \in \mathbb{N}$, we obtain

$$e^{(\lambda-a+c)\pi+2cs} \leq De^{\varepsilon s},$$

which is impossible for $\varepsilon < 2c$. In the second case, there exist $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$e^{-a(t-s)}T^1(t,s) \leq De^{-\lambda(s-t)+\varepsilon|s|} \quad \text{for } t \leq s.$$

Taking $t = 2l\pi$ and $s = (2l+1)\pi$ with $l \in \mathbb{N}$, we obtain

$$e^{(a-c+\lambda)\pi+2cs} \leq De^{\varepsilon s},$$

which again is impossible for $\varepsilon < 2c$. In other words, for each $a \in \mathbb{R}$ the evolution family $(T^1)_a(t,s)$ does not admit a nonuniform dichotomy. Thus, $\Sigma = \mathbb{R}$.

Example 3.3. Take numbers

$$b_1 \geq a_1 > b_2 \geq a_2 > b_3 \geq a_3 > \cdots > b_k \geq a_k$$

for some integer $k \geq 1$. For each $j \in \{1,\ldots,k\}$, let $\phi_j \colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\phi_j(t) = a_j$ for $t \leq -1$ and $\phi_j(t) = b_j$ for $t \geq 1$. We consider a linear equation $x' = A(t)x$ on X, where $A(t)e_j = a_j(t)e_j$ for each j, taking

$$a_j(t) = \begin{cases} \phi_j(t) + \frac{1}{2\sqrt{1+t}}\sin t + \sqrt{1+t}\cos t, & t \geq 0, \\ \phi_j(t) - \frac{1}{2\sqrt{1-t}}\sin t + \sqrt{1-t}\cos t, & t < 0 \end{cases}$$

for $1 \leq j \leq k$ and $a_j(t) = -3t^2$ for $j > k$. The corresponding evolution family $T(t,s)$ satisfies $T(t,s)e_j = T^j(t,s)e_j$ for each j, where

$$T^j(t,s) = \begin{cases} e^{b_j(t-s)+\sqrt{1+t}\sin t - \sqrt{1+s}\sin s}, & t,s \geq 0, \\ e^{b_jt - a_js + \sqrt{1+t}\sin t - \sqrt{1+|s|}\sin s}, & t \geq 0, s < 0, \\ e^{a_j(t-s)+\sqrt{1+|t|}\sin t - \sqrt{1+|s|}\sin s}, & t,s < 0 \end{cases} \tag{3.1}$$

for $1 \leq j \leq k$ and $T^j(t,s) = e^{t^3-s^3}$ for $j > k$.

We claim that for each $j \in \{1,\ldots,k\}$ and $a \notin [a_j, b_j]$, the evolution family $(T^j)_a(t,s)$ admits a nonuniform dichotomy. Take $a > b_j$. Since $a_j \leq b_j$, we have

$$e^{-a(t-s)}T^j(t,s) \leq e^{-(a-b_j)(t-s)+\sqrt{1+|t|}+\sqrt{1+|s|}} \tag{3.2}$$

for $t \geq s$. Moreover, since

$$\frac{\sqrt{1+|t|}}{|t|} \to 0 \quad \text{when } |t| \to +\infty,$$

given $\delta > 0$, there exists $D = D(\delta) > 0$ such that

$$e^{\sqrt{1+|t|}} \leq De^{\delta|t|} \quad \text{for } t \in \mathbb{R}.$$

Hence, it follows from (3.2) that

$$\begin{aligned} e^{-a(t-s)}T^j(t,s) &\leq D^2 e^{-(a-b_j)(t-s)+\delta|t|+\delta|s|} \\ &\leq D^2 e^{-(a-b_j-\delta)(t-s)+2\delta|s|} \end{aligned}$$

for $t \geq s$. Since $a - b_j > 0$ and δ is arbitrary, this shows that $(T^j)_a(t,s)$ admits a nonuniform dichotomy with projections $P_t = \mathrm{Id}$. Similarly, for $a < a_j$ and $t \leq s$, we have

$$e^{-a(t-s)}T^j(t,s) \leq D^2 e^{(a_j-a-\delta)(t-s)+2\delta|s|}.$$

Hence, $(T^j)_a(t,s)$ admits a nonuniform dichotomy with projections $P_t = 0$. We also show that for each $j \in \{1,\dots,k\}$ and $a \in [a_j, b_j]$, the evolution family $(T^j)_a(t,s)$ does not admit a nonuniform dichotomy. Since $b_j - a \geq 0$, the first branch of

$$(T^j)_a(t,s) = \begin{cases} e^{(b_j-a)(t-s)+\sqrt{1+t}\sin t-\sqrt{1+s}\sin s}, & t,s \geq 0, \\ e^{(b_j-a)t-(a_j-a)s+\sqrt{1+t}\sin t-\sqrt{1+|s|}\sin s}, & t \geq 0, s < 0, \\ e^{(a_j-a)(t-s)+\sqrt{1+|t|}\sin t-\sqrt{1+|s|}\sin s}, & t,s < 0 \end{cases}$$

precludes the existence of a nonuniform dichotomy with projections $P_t = \mathrm{Id}$. Moreover, since $a_j - a \leq 0$, the third branch precludes the existence of a nonuniform dichotomy with projections $P_t = 0$. We conclude that for each $j \in \{1,\dots,k\}$, the evolution operator $(T^j)_a(t,s)$ admits a nonuniform dichotomy if and only if $a \notin [a_j, b_j]$. On the other hand, for each $j > k$ and $a \in \mathbb{R}$, the evolution family $(T^j)_a(t,s)$ admits a nonuniform dichotomy with projections $P_t = \mathrm{Id}$.

Finally, we show that $\Sigma = \bigcup_{j=1}^k [a_j, b_j]$. Take $a \in \mathbb{R} \setminus \bigcup_{j=1}^k [a_j, b_j]$. From what is proved, it follows that for $a > b_1$ the evolution family $T_a(t,s)$ admits a nonuniform dichotomy with projections $P_t = \mathrm{Id}$. Moreover, for $a < a_k$ it admits a nonuniform dichotomy with projections P_t given by

$$P_t e_j = 0 \text{ for } 1 \leq j \leq k \quad \text{and} \quad P_t e_j = e_j \text{ for } j > k.$$

Finally, take $j \in \{1,\dots,k-1\}$ such that $a_j > a > b_{j+1}$. Then $T_a(t,s)$ admits a nonuniform dichotomy with projections P_t given by $P_t e_i = 0$ for $1 \leq i \leq j$ and $P_t e_i = e_i$ for $i > j$. Therefore, $\Sigma \subset \bigcup_{j=1}^k [a_j, b_j]$. Conversely, take $a \in [a_j, b_j]$ for some $j = 1,\dots,k$ and assume that $a \notin \Sigma$. Since $T_a(t,s)$ admits a nonuniform dichotomy, the same happens to $(T^j)_a(t,s)$, but this is impossible since $a \in [a_j, b_j]$.

A similar construction can be effected for the case when the spectrum has unbounded connected components.

Example 3.4. Take numbers a_n and b_n as in (2.6) with $\lim_{j\to+\infty} a_j = -\infty$. We consider the evolution family $T(t,s)$ given by $T(t,s)e_j = T^j(t,s)e_j$ for $j \in \mathbb{N}$ with $T^j(t,s)$ as in (3.1). Proceeding as in Example 3.3, one can show that for each $a > b_1$ the evolution family $T_a(t,s)$ admits a nonuniform dichotomy with projections $P_t = \mathrm{Id}$. Moreover, for $a \in (b_{j+1}, a_j)$ with $j \in \mathbb{N}$ it admits a nonuniform dichotomy with projections P_t given by

$$P_t e_i = 0 \text{ for } 1 \le i \le j \quad \text{and} \quad P_t e_i = e_i \text{ for } i \ge j+1.$$

Finally, in a similar manner to that in Example 3.3, we have $[a_j, b_j] \subset \Sigma$ for each $j \in \mathbb{N}$ and so $\Sigma = \bigcup_{n=1}^{\infty} [a_n, b_n]$. A similar construction can be effected for the case when $I_1 = [a_1, +\infty)$.

Example 3.5. Take numbers a_n and b_n as in (2.6) with $\lim_{j \to +\infty} a_j = a_\infty \in \mathbb{R}$. For each $n \in \mathbb{N}$, let $\phi_n \colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\phi_n(t) = a_n$ for $t \le -1$ and $\phi_n(t) = b_n$ for $t \ge 1$. We consider the linear equation $x' = A(t)x$ on X, where $A(t)e_j = a_j(t)e_j$ and

$$a_j(t) = \begin{cases} \phi_j(t) + \frac{1}{2\sqrt{1+t}} \sin t + \sqrt{1+t} \cos t, & t \ge 0, \\ \phi_j(t) - \frac{1}{2\sqrt{1-t}} \sin t + \sqrt{1-t} \cos t, & t < 0 \end{cases}$$

for $j \in \mathbb{N}$. The corresponding evolution family $T(t,s)$ satisfies $T(t,s)e_j = T^j(t,s)e_j$, for $j \in \mathbb{N}$, with $T^j(t,s)$ as in (3.1). Proceeding as in Example 3.3, one can show that for each $a > b_1$ the evolution family $T_a(t,s)$ admits a nonuniform dichotomy with projections $P_t = \mathrm{Id}$. Moreover, for $a \in (b_{j+1}, a_j)$ with $j \in \mathbb{N}$ it admits a nonuniform dichotomy with projections P_t given by

$$P_t e_i = 0 \text{ for } 1 \le i \le j \quad \text{and} \quad P_t e_i = e_i \text{ for } i \ge j+1.$$

As in Example 3.3, we have $[a_j, b_j] \subset \Sigma$ for each $j \in \mathbb{N}$. Finally, by Lemma 2.5, $(-\infty, a_\infty] \subset \Sigma$ and so $\Sigma = \bigcup_{n=1}^{\infty} [a_n, b_n] \cup (-\infty, a_\infty]$. Again, a similar construction can be effected for the case when $I_1 = [a_1, +\infty)$.

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